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THE ASSOCIATED PRIMES OF $H_G^*(X)$

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The cohomology ring of the title is the Borel equivariant cohomology ring $H^*(EG\times^G X, \mathbb{Z}/p\mathbb{Z})$ of a G-space X with $\mathbb{Z}/p\mathbb{Z}$ coefficients (p is a fixed prime). Here, G is a compact Lie group and $EG\to BG$ is a classifying bundle for principal G-bundles; $EG\times^G X$ is the orbit space of $EG\times X$ under the diagonal action of G. We refer the reader to [4] for the assumptions we make on X, and for a list of the properties of $H_G^*(X)$ that we use in this paper.

The main result of this paper is Theorem 3.1 which states that the associated primes of $H_G^*(X)$ are invariant under the Steenrod operations; actually it is somewhat stronger and shows that the associated primes are all p-toral, i.e., they can be obtained by restricting the cohomology ring $H_G^*(X)$ to the cohomology ring of a p-torus (a product of cyclic groups of order p).

An application to the cohomology of groups is given in Section 4.

1. A comparison theorem

Suppose that G is a compact Lie group acting on the left on a space M. Fix an embedding of G as a closed subgroup of another compact Lie group U. Let S be a closed subgroup of U. One has three spaces associated with this data:

(i) the S-space $G/M\times U$ – $G/M\times U$ is the orbit space of the action $(m,u)\mapsto(gm,gu)$ of G on $M\times U$, and the S-action is given by $G(m,u)\mapsto G(m,us^{-1})$.

(ii) the $G\times S$-space $M\times U$ – the $G\times S$ action is given by $(g,s)(m,u)=(gm,gus^{-1})$.

(iii) the G-space $M\times U/S$ – U/S is the orbit space of the action $u\mapsto us^{-1}$ of S on U; the G-action is given by $(m,us)\mapsto(gm,gus^{-1})$.

We see that the orbit spaces $(G/(M\times U))/S$, $(M\times U)/(G\times S)$, and $(M\times U/S)/G$ are all homeomorphic, we denote this common orbit space by $G/(M\times U)/S$. Denote the orbit projections by π_S, $\pi_{G\times S}$ and π_G, respectively.

There are equivariant maps

$$(S, G/(M\times U)) \xleftarrow[\theta_S]{} (G\times S, M\times U) \xrightarrow[\theta_G]{} (G, M\times U/S)$$

where $\theta_S = (\mathrm{pr}_S, \varrho_G)$, $\theta_G = (\mathrm{pr}_G, \mathrm{id}_M \times \varrho_S)$. ($\mathrm{pr}_S$ and pr_G are projections and ϱ_G, ϱ_S are orbit space projections.)

If X, Y and Z are open or closed invariant subspaces of $G/(M\times U)$, $M\times U$, and $M\times U/S$, respectively, such that $\theta_S^{-1}(X) = Y = \theta_G^{-1}(Z)$ then $\pi_S(X) = \pi_{G\times S}(Y) = \pi_G(Z)$ in $G/(M\times U)/S$; also, we have $\pi_S(X) = X/S$, $\pi_{G\times S}(Y) = Y/(G\times S)$ and $\pi_G(Z) = Z/G$ (i.e. all the various topologies here that one can compare are identical).

Proposition 1.1. *The maps θ_S, θ_G induce ring isomorphisms on equivariant cohomology*

$$H_S^*(G/(M\times U)) \xrightarrow[\theta_S^*]{\cong} H_{G\times S}^*(M\times U) \xleftarrow[\theta_G^*]{\cong} H_G^*(M\times U/S)$$

(here the cohomology may have coefficients in any fixed commutative ring). More generally, if X, Y, Z are open or closed invariant subspaces of $G/(M\times U)$, $M\times U$ and $M\times U/S$, respectively, such that $\theta_S^{-1}(X) = Y = \theta_G^{-1}(Z)$ then there are isomorphisms of equivariant cohomology rings

$$H_S^*(X) \xrightarrow[\theta_S^*|Y]{\cong} H_{G\times S}^*(Y) \xleftarrow[\theta_G^*|Y]{\cong} H_G^*(Z).$$

Proof. There are various ways to prove this. We use Leray spectral sequences, and prove the more general statement.

One has a commutative diagram

$$\begin{array}{ccccc} ES\times^S X & \xleftarrow{\ \theta_S\ } & (EG\times ES)\times^{G\times S} Y & \xrightarrow{\ \theta_G\ } & EG\times^G Z \\ & {\scriptstyle \pi_S}\searrow & \downarrow{\scriptstyle \pi_{G\times S}} & \swarrow{\scriptstyle \pi_G} & \\ & & W = X/S = Y/(G\times S) = Z/G \subseteq G/(M\times U)/S & & \end{array}$$

It should be clear what the maps are.

One gets homomorphisms of the Leray spectral sequences associated to π_S, $\pi_{G\times S}$, π_G:

$$\begin{array}{c} E_2^{s,t} = H^s(W, \mathscr{H}_S^t) \Rightarrow H_S^{s+t}(X) \\ \Big\downarrow {\scriptstyle \theta_S^*} \\ E_2^{s,t} = H^s(W, \mathscr{H}_{G\times S}^t) \Rightarrow H_{G\times S}^{s+t}(Y) \\ \Big\uparrow {\scriptstyle \theta_G^*} \\ E_2^{s,t} = H^s(W, \mathscr{H}_G^t) \Rightarrow H_G^{s+t}(Z) \end{array}$$

Here, $\mathscr{H}_Q^t$ ($Q=G$, S or $G\times S$) is the sheaf on W associated to the presheaf on $W: V \to H_Q^t(\pi_Q^{-1}(V))$.

The stalks of $\mathscr{H}_Q^t$ are $\mathscr{H}_{Q,\xi}^t = H_Q^t(\pi_Q^{-1}(\xi)) = H_{Q_\xi}^t$ where $\xi = G(m,u)S \in W$ and

$$Q_\xi = \begin{cases} S_{G(m,u)} = \{s\in S \mid G(m,us^{-1}) = G(m,u)\}, & Q=S,\\ (G\times S)_{(m,u)} = \{(g,s) \mid gus^{-1}=u, gm=m\}, & Q=G\times S,\\ G_{(m,uS)} = \{g\in G \mid guS=uS, gm=m\}, & Q=G. \end{cases}$$

Since there are *isomorphisms*

$$S_{G(m,u)} \xleftarrow[\mathrm{pr}_S]{\cong} (G\times S)_{(m,u)} \xrightarrow[\mathrm{pr}_G]{\cong} G_{(m,uS)}$$

the sheaves $\mathscr{H}_Q^t$ for $Q=G$, S or $G\times S$, are all isomorphic; this means that the E_2 terms, and hence the abutments, of the three spectral sequences are isomorphic. □

In principle, this proposition allows one to specialize from general group actions to toral actions. We make use of this specialization in the following two sections.

For later use, we note that the isomorphism of the proposition is an isomorphism of modules over the Steenrod algebra.

2. The p-rank filtration

In this section we assume that G acts smoothly on a differentiable manifold M, and that p is either a fixed prime or is zero.[1] If $p>0$ and H is an closed subgroup of G, define the *p-rank of H*, $\mathrm{rk}_p H$, as the maximum rank of a p-torus in H. If $p=0$, define $\mathrm{rk}_0 H$ as the maximum rank of a 0-torus in the connected component of the identity H_0 in H.

If n is the p-rank of G, then there is a filtration of M, the *p-rank filtration*,

$$M = M_0 \geq M_1 \geq \cdots \geq M_n \geq M_{n+1} = \emptyset,$$

defined by $M_i = \{x\in M \mid \mathrm{rk}_p G_x \geq i\}$. These subspaces are *invariant* (since $G_{hx} = hG_xh^{-1}$ for $h\in G$). They are also *closed*: any $x\in M$ has an open neighborhood U such that G_u is subconjugate to G_x for every $u\in U$; thus $\mathrm{rk}_p G_u \leq \mathrm{rk}_p G_x$ for every $u\in U$.

Thus there is a filtration

$$0 = F_{n+1} \leq F_n \leq \cdots \leq F_1 \leq F_0 = H_G^*(M)$$

defined by $F_i = \ker(H_G^*(M) \to H_G^*(M-M_i))$. Call this filtration the *p-rank filtration* on $H_G^*(M)$.

We define subspace $M_{(i)}$ of M by $M_{(i)} = \{x\in X \mid \mathrm{rk}_p G_x = i\}$.

[1] If $p=0$, we assume that the cohomology ring has rational coefficients.

Proposition 2.1. *If all of the isotropy groups G_x of the G-action on M are p-tori, and if they fall into a finite number of conjugacy classes then $M_{(i)}$ is a smooth closed G-invariant submanifold of $M-M_{i+1}$, for $i=0,1,\dots,n$.*

Proof. The proof is elementary; we give it to make clear exactly why all the various hypotheses are needed.

In any case $M_{(i)}$ is G-invariant and closed in $M-M_{i+1}$ since $M_{(i)}=M_i-M_{i+1}=M_i\cap(M-M_{i+1})$.

Let $S_i=\{[G_x]\mid \mathrm{rk}_p G_x=i\}$ for each i. (Here, $[H]$ denotes the conjugacy class of the subgroup H.)

We claim that

$$M_{(i)}=\bigcup_{[H]\in S_i} G(M-M_{i+1})^H \tag{$*$}$$

and that this union is disjoint. In the equation (*),

$$G(M-M_{i+1})^H=\{gx\mid x\in(M-M_{i+1})^H, g\in G\}.$$

We assume $p<0$. The proof for $p=0$ is similar.

Let $x\in M_{(i)}$. Then $[G_x]\in S_i$, and certainly $x\in G(M-M_{i+1})^{G_x}$. On the other hand, if $x\in G(M-M_{i+1})^H$ where $[H]\in S_i$, then $x=gz$, $z\in(M-M_{i+1})^H$, $g\in G$. Thus $\mathrm{rk}_p G_x\le i$ and $H\le G_z$, so $i=\mathrm{rk}_p H\le \mathrm{rk}_p G_z\le i$, so $\mathrm{rk}_p G_z=i$. Since $G_x=gG_zg^{-1}$, $\mathrm{rk}_p G_x=i$, so $x\in M_{(i)}$. Thus equality holds in (*). (Note that we did not need any hypotheses about the isotropy groups here.)

Now, for disjointness, assume that $x\in G(M-M_{i+1})^H\cap G(M-M_{i+1})^K$, where $[H]$, $[K]\in S_i$ and $[H]\ne[K]$. Then $x=gz=g'z'$ where $z\in(M-M_{i+1})^H$ and $z'\in(M-M_{i+1})^K$ and $g,g'\in G$. Thus, $H\le G_{z'}$, $K\le G_{z'}$ and $G_x=gG_zg^{-1}=g'G_{z'}(g')^{-1}$. Let $\Gamma=\langle gHg^{-1},g'K(g')^{-1}\rangle\le G_x$ be the subgroup generated by gHg^{-1} and $g'K(g')^{-1}$. We claim $\mathrm{rk}_p\Gamma>i$, contradicting $\mathrm{rk}_p G_x=i$; since $[H]\ne[K]$, $gHg^{-1}\ne g'K(g')^{-1}$. Thus there is an element ξ in, say gHg^{-1}, not in $g'K(g')^{-1}$. It is at this point that we require that all isotropy be p-toral. For, since gGg^{-1} is p-toral, ξ is of order p, and commutes with $g'K(g')^{-1}$. So $\langle g'K(g')^{-1},\xi\rangle$, the subgroup of Γ generated by $g'K(g')^{-1}$ and ξ must have p-rank strictly greater than the p-rank of $g'K(g')^{-1}$, which is equal to i. Thus the claim about equation (*) is proved.

Now, if $[H]\in S_i$, then $G(M-M_{i+1})^H$ is the set of points in $M-M_{i+1}$ on orbits of type G/H (i.e., the points whose isotropy groups are conjugate to H), so it is a smooth submanifold of $M-M_{i+1}$ (see, e.g., [2]). Since $M_{(i)}$ is a *finite* (here, finiteness of S_i is used) disjoint union of submanifolds, it, too, is a submanifold. □

We return to the notation of Section 1, i.e., G and S are closed subgroups of some large compact Lie group U. We also assume that $p>0$.

Consider the maps

$$\begin{array}{ccccc} G/(M\times U) & \xleftarrow[\theta_S]{} & M\times U & \xrightarrow[\theta_G]{} & M\times U/S \\ \| & & \| & & \| \\ MS & & MU & & MG \end{array}$$

(Note that MS, MU and MG are all smooth S, $G\times S$ and G (respectively) manifolds.)

Since the isotropy groups

$$S_{G(m,u)} \xleftarrow[\cong]{} (G\times S)_{(m,u)} \xrightarrow[\cong]{} G_{(m,uS)}$$

are isomorphic for $(m,u)\in M\times U$, we see that (using the notation of Section 2)

(i) $$\theta_S^{-1}((G/(M\times U)_i) = (M\times U)_i = \theta_G^{-1}((M\times U/S)_i)$$

so also

$$\theta_S^{-1}(MS-MS_i) = MU-MU_i = \theta_G^{-1}(MG-MG_i)$$

and

(ii) $$\theta_S^{-1}(MS_{(i)}) = MU_{(i)} = \theta_G^{-1}(MG_{(i)}).$$

Proposition 2.2. *If* $F.(S)$, $F.(G\times S)$ *and* $F.(G)$ *denote the p-rank filtrations on* $H_S^*(MS)$, $H_{G\times S}^*(MU)$ *and* $H_G^*(MG)$, *respectively, then*

$$\theta_S^{*-1}(F.(G\times S)) = F.(S) \quad \textit{and} \quad \theta_G^{*-1}(F.(G\times S)) = F.(G).$$

Proof. One has for each i a commutative diagram

$$\begin{array}{ccc} H_S^*(MS) & \xrightarrow{\text{res}} & H_S^*(MS-MS_i) \\ \downarrow \theta_S^* \cong & & \downarrow \theta_S^* \cong \\ H_{G\times S}^*(MU) & \xrightarrow{\text{res}} & H_{G\times S}^*(MU-MU_i) \\ \uparrow \theta_G^* \cong & & \uparrow \theta_S^* \cong \\ H_G^*(MG) & \xrightarrow{\text{res}} & H_G^*(MG-MG_i) \end{array}$$ □

Let us now assume that U is a unitary group and that S is the diagonal p-torus of U. Then all isotropy groups of the G-action on $M\times U/S$ are p-toral [8] so that $MG_{(i)}$ and $MS_{(i)}$ are smooth submanifolds of $MG-MG_{i+1}$ and $MS-MS_{i+1}$, respectively (Proposition 2.1).

In fact, $MS_{(i)}$ is an *S-orientable* submanifold of $MS-MS_{i+1}$ (see, e.g., [4]). Thus one has an exact Gysin triangle

$$\begin{array}{ccc} & H_S^*(MS-MS_{i+1}) & \\ {\scriptstyle \Phi_S}\nearrow & & \searrow{\scriptstyle \text{res}} \\ H_S^*(MS_{(i)}) & \longleftarrow & H_S^*(MS-MS_i) \end{array}$$

We define a 'fake' Gysin map

$$\Phi_G : H_G^*(MG_{(i)}) \to H_G^*(MG-MG_{i+1})$$

to be

$$\Phi_G = \theta_G^{*-1}\theta_S^*\Phi_S\theta_S^{*-1}\theta_G^*.$$

Proposition 2.3. (Compare with Theorem 1 of [4].) *If p is odd, and U, S are as in the previous paragraph, then*

(a) *there is a fake exact Gysin triangle*

$$\begin{array}{ccc} & H_G^*(MG-MG_{i+1}) & \\ {\scriptstyle \Phi_G}\nearrow & & \searrow{\scriptstyle \text{res}} \\ H_G^*(MG_{(i)}) & \overset{\text{res}}{\longleftarrow} & H_G^*(MG-MG_i) \end{array}$$

with Φ_G injective, thus yielding

(b) $$F_i(G)/F_{i+1}(G) \cong H_G^*(MG_{(i)})$$

as k-modules.

Proof. Use the fake Gysin map above; the cohomology proposition and (i) and (ii) above; and the results of [4]. □

Question. Is there always a *real* Gysin map Φ_G? Or, in other words, is $MG_{(i)}$ always a G-orientable submanifold of $MG - MG_{i+1}$?

3. Steenrod operations and Ass($H_G^*(M)$)

In this section, all cohomology has coefficients in $k = Z/pZ$ where $p > 0$. As in Section 2, we assume that M and the action of G on M are smooth. Imbed G in a unitary group U, and let S be the diagonal p-torus of U. Let

$$H_G(M) = \begin{cases} H_G^{\text{ev}}(M), & p \text{ odd}, \\ H_G^*(M), & p = 2. \end{cases}$$

$H_G^*(M)$ and $H_G(M)$ are algebras over the commutative ring $H_G(pt)=H_G$. If $H^*(M)$ is finitely generated over Z/pZ (as a module), then $H_G^*(M)$ and $H_G(M)$ are finitely generated as modules over H_G [5], [9]; we assume this from now on. In this case, in fact, since H_G is a noetherian ring, so are $H_G^*(M)$ and $H_G(M)$ [5], [9].

In general, if R is a Noetherian ring and E is an R-module, one defines $\mathrm{Ass}_R E$ to be the set of primes $\mathfrak{p}$ in R such that E contains a submodule isomorphic with $R/\mathfrak{p}$. We denote $\mathrm{Ass}_R R$ simply by $\mathrm{Ass}\, R$.

Theorem 3.1. *If p is odd, then the prime ideals in* $\mathrm{Ass}(H_G(M))$ *are invariant under the Steenrod p-th power operations.*

Proof. We may assume that G is a p-torus by the following argument.

Section 1 gives an isomorphism $H_G(MG)\cong H_S(MS)$. This isomorphism is induced by maps at the space level, so it is an isomorphism of rings and of modules over the Steenrod algebra.

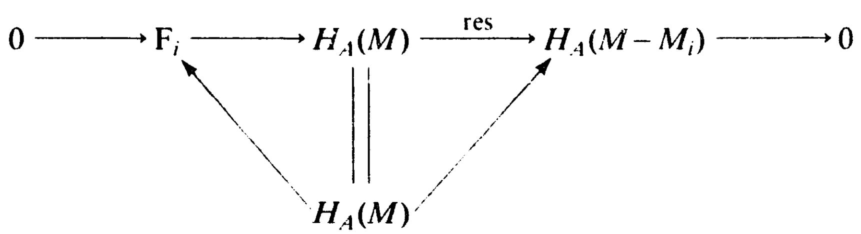
Assuming that the theorem is true for p-tori, we see that the primes in $\mathrm{Ass}(H_S(MS))$ are invariant under the Steenrod operations. So the primes in $\mathrm{Ass}(H_G(MG))$ are invariant under the Steenrod operations.

Quillen [8] shows that there is an injection $H_G(M)\to H_G(MG)$ (induced by the projection $M\times U/S\xrightarrow{\mathrm{pr}} M$) making $H_G(MG)$ into a free, and hence faithfully flat, extension of $H_G(M)$. Thus (see, e.g., [7]),

$$\{\mathrm{pr}^{*-1}(\mathfrak{p})\mid \mathfrak{p}\in \mathrm{Ass}\, H_G(MG)\}=\mathrm{Ass}\, H_G(M).$$

Since pr* respects the Steenrod operations, we see that the primes in $\mathrm{Ass}\, H_G(M)$ are invariant under the Steenrod operations.

So, we now assume that $G=A$ is a p-torus of rank n. Let F be the p-rank filtration of $H_A(M)$. Proposition 2.3 (or Theorem 1 of [4]) shows that $\mathrm{F}_i/\mathrm{F}_{i+1}\cong H_A(M_{(i)})$ as k-modules for each i. It is necessary to point out that this is actually an isomorphism of $H_A(M)$-modules. Here, $H_A(M_{(i)})$ is considered as an $H_A(M)$-module via restriction, and F_i is considered as an $H_A(M)$-module via restriction also.



To see that one has an $H_A(M)$-isomorphism, we note that the k-isomorphism results from the following commutative diagram with exact rows and columns (see [4]):

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F_{i+1} & \longrightarrow & F_i & \longrightarrow & F_i/F_{i+1} & \longrightarrow & 0 \\
 & & \cap & & \cap & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H_A(M) & & H_A(M) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 \longrightarrow H_A(M_{(i)}) & \xrightarrow{\Phi_A} & H_A(M-M_{i+1}) & \longrightarrow & H_A(M-M_i) & \longrightarrow & 0 & &
\end{array}
$$

Here Φ_A is the Gysin map associated with the embedding $M_{(i)} \to M - M_{i+1}$. One has that $\Phi_A(\operatorname{res}\xi \cdot \eta) = \xi \cdot \Phi_A(\eta)$ for ξ in $H_A(M-M_{i+1})$; thus, by transitivity of restriction we see that Φ_A respects the $H_A(M)$-module structure on $H_A(M_{(i)})$. (One should point that Φ_A is really defined from $H_A^*(M_{(i)}) \to H_A^*(M-M_{i+1})$, not from $H_A(M_{(i)})$, but one knows that the Gysin map, while not respecting degree, respects parity in this case (again see [4] for details).) Certainly all the other maps respect the $H_A(M)$-module structure, so the desired result follows.

Therefore (see, e.g., [7]),

$$\operatorname{Ass} H_A(M) \subseteq \bigcup_i \operatorname{Ass}_{H_A(M)}(F_i/F_{i+1}) = \bigcup_i \operatorname{Ass}_{H_A(M)} H_A(M_{(i)}).$$

We calculate $\operatorname{Ass}_{H_A(M)} H_A(M_{(i)})$. Since $M_{(i)} = \bigcup_B (M - M_{i+1})^B$ (see Section 2), we write

$$H_A(M_{(i)}) = \bigoplus_B H_A((M-M_{i+1})^B)$$

where B runs over the subtori B of A of rank i. This equality, of course, preserves the $H_A(M)$-module structure.

Let $\{c\}$ be the set of connected components of $(M-M_{i+1})^B$ and let $\mathscr{C} = \{\bar{c} = \bigcup_{a \in A} ac\}$ be the set of 'A-invariant components' of $(M-M_{i+1})^B$ (this set is finite; see, e.g., Lemma 2 of [4]). Then $H_A((M-M_{i+1})^B) = \bigoplus_{\mathscr{C}} H_A(\bar{c})$, so finally we have

$$\operatorname{Ass}_{H_A(M)} H_A(M_{(i)}) = \bigcup_B \operatorname{Ass}_{H_A(M)} H_A((M-M_{i+1})^B) = \bigcup_B \bigcup_{\mathscr{C}} \operatorname{Ass}_{H_A(M)} H_A(\bar{c}).$$

The point of all this is that one knows exactly what $\operatorname{Ass}_{H_A(M)} H_A(\bar{c})$ is. Namely,

$$\operatorname{Ass}_{H_A(M)}(H_A(\bar{c})) = \{\mathfrak{p}_{(B,\Gamma)}\}$$

where

$$\mathfrak{p}_{(B,\Gamma)} = \ker(H_A(M) \to H_B(\mathrm{pt})/\sqrt{0})$$

and $\mathrm{pt} \in \Gamma$ – the component of M^B containing c (we adopt the notation of [8], Part II).

How does one see this? First, there is only one minimal prime of $H_A(\bar{c})$: the prime $\mathfrak{p}_B = \ker(H_A(\bar{c}) \to H_B(\mathrm{pt})/\sqrt{0})$ (where pt is any point in $\bar{c}$). This follows from Quillen's characterization of the minimal primes of an equivariant cohomology ring [8] since

(i) $\bar{c}$ is a space with only one isotropy group B with respect to the A-action, and

(ii) $\bar{c}$ has a transitive A-action on its components.

Second, there are no *embedded associated primes in* $H_A(\bar{c})$ since $H_A(\bar{c})$ is 'Cohen-Macaulay' – this follows from [3] since $\bar{c}$ also satisfies $\bar{c}^B = \bar{c}$. Thus $\mathrm{Ass}\, H_A(\bar{c}) = \{\mathfrak{p}_B\}$, $\mathfrak{p}_B$ as above.

It is not hard to see (see, e.g., [7]) that this means that $\mathrm{Ass}_{H_A(M)} H_A(\bar{c}) = \{\theta^{-1}(\mathfrak{p}_B)\}$ where $\theta : H_A(M) \to H_A(\bar{c})$ is the restriction map defining $H_A(\bar{c})$ as an $H_A(M)$-module. Now, $\theta^{-1}(\mathfrak{p}_B) = \mathfrak{p}_{(B,\Gamma)}$; clearly,

$$\mathrm{Ass}_{H_A(M)} H_A(\bar{c}) \subseteq \mathfrak{p}_{(B,\Gamma)} \subseteq \theta^{-1}(\mathfrak{p}_B).$$

Also, $\theta^{-1}(\mathfrak{p}_B)$ is *minimal* over $\mathrm{Ass}_{H_A(M)} H_A(\bar{c})$ since it is the only associated prime.

All this shows that

$$\mathrm{Ass}\, H_A(M) \subseteq \{\mathfrak{p}_{(B,\Gamma)} \mid B \subseteq A,\ \Gamma \text{ is a component of } M^B\}.$$

But it is clear that the primes $\mathfrak{p}_{(B,\Gamma)}$ are all the invariant under the Steenrod operations, being defined by maps at the space level. □

Corollary 3.2. *With the hypotheses of the above theorem, the prime ideals in* $\mathrm{Ass}\, H_G(M)$ *are all toral, i.e., they are of the form* $\mathfrak{p}_{(B,c)} = \ker(H_G(M) \to H_B(\mathrm{pt})/\sqrt{0})$. *Here,* $\mathrm{pt} \in c$ *where c is a component of M^B and B is a p-torus in G.*

Proof. One knows (e.g., see [7]) that all associated primes are graded, and by the theorem, they are invariant under the Steenrod operations. Thus, by [8] they are all toral. □

We point out that the results of [8], Part I, alone show that the minimal primes are invariant under the Steenrod operations (since they are all toral). The point of the theorem is that the embedded primes are toral, too. Also, the corollary could be deduced (without appealing to the results of [8]) by the *proof* of the theorem.

Remarks. (i) The results of this section should be true for $p = 2$ as well. We assume $p \neq 2$ because the results in [4] that the proof of the theorem depend on are given only for $p \neq 2$.

(ii) Theorem 3.1 can be restated in the following form.

Theorem 3.3. *Let G and M be as in Theorem 3.2. Let* $\mathscr{P} : H_G(M) \to H_G(M)[t]$ *be the map given by the Steenrod operations, i.e.,*

$$\mathscr{P}(r) = \sum_{i=0}^{\infty} \mathscr{P}^i(r) t^i$$

where $r \in H_G(M)$. Then $\mathfrak{p}$ *is a nondegenerate homomorphism of noetherian rings.* (For definition of nondegenerate homomorphism, see [7].)

Now, every flat homomorphism of (noetherian) rings is nondegenerate [7] (although the converse is not true in general). A question here is: "Is $\mathscr{P}$ flat?"

In the case where $G = A$ is a p-torus, and $M = \text{pt}$ is a point, the map

$$H_A/\sqrt{0} \xrightarrow{\mathscr{P}} H_A/\sqrt{0}[t], \qquad k[x_1, \ldots, x_n] \xrightarrow{\mathscr{P}} k[x_1, \ldots, x_n][t]$$

(where $\deg x_i = 2$) is given by

$$\mathscr{P}(x_i) = x_i + x_i^p(t).$$

This map is *smooth* (and hence flat).

(iii) We note that Theorem 3.2 implies that if $a \in H_G(M)$ is a zero-divisor then $\mathscr{P}^i(a)$ is also a zero-divisor for every $i \geq 0$ (although this fact should be proved in a more straightforward manner without using the methods of Theorem 3.2).

4. An application to the cohomology of groups

Let G be a *finite* group; all other notation is carried over from the previous sections. Let A be a p-torus, and let $\text{reg}\, A$ be the regular representation (complex) of A. The Chern class $c_j(\text{reg}\, A)$ lies in $H_A^{2j} = H_A^{2j}(\text{pt})$ (where pt is a one-point space) for $j = 1, 2, \ldots, p^n$ where the order of A is p^n. One knows that

(i) $c_j(\text{reg}\, A) = 0$ unless $j = p^n - p^i$, $i = 0, 1, \ldots, n-1$;

(ii) $c_{p^n-1}(\text{reg}\, A), c_{p^n-p}(\text{reg}\, A), \ldots, c_{p^n-p^{n-1}}(\text{reg}\, A)$ generate a polynomial subring of H_A; and

(iii) if B is a nontrivial subgroup of A of index p^{n-i} then

$$\text{res}_B(c_{p^n-p^{n-1}}(\text{reg}\, A)) = c_{p^i-p^{i-1}}(\text{reg}\, B)^{p^{n-i}}$$

(see, e.g., [8], Part II).

Theorem 4.1. *If $H_G = H_G(\text{pt})$ and $p \mid |G|$, then there is a non-zero-divisor in H_G so that* $\text{depth}\, H_G \geq 1$.

Proof. The set of zero-divisors in H_G is equal to the union of the associated primes of H_G [6]. By the main theorem, these associated primes are all toral. Then there exists an element γ of H_G whose restriction to H_A for *every* p-torus A in G is not zero (thus γ can't be in any toral prime, so γ can't be a zero-divisor in G). One sees this as follows: for each p-torus A in G, let

$$\gamma_A = [c_{p^{\text{rk}\, A} - p^{\text{rk}\, A - 1}}(\text{reg}\, A)]^{p^{d - \text{rk}\, A}}$$

where $d = \dim H_G = \max\{\operatorname{rk} A \mid A \text{ is a } p\text{-torus in } G\}$. Then if B is subconjugate to A, one sees that the map $H_A \to H_B$ takes γ_A to γ_B. Thus by Corollary 2.6 of [1], one has an element γ and a power q of p such that

$$\operatorname{res}_A(\gamma) = \gamma_A^q$$

for every p-torus A in G. Certainly γ_A^q is not zero in H_A so γ is not in any toral prime. Hence γ is not a zero-divisor in H_G. □

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